Answers to questions in the

Written Exam at the Department of Economics winter 2017-18

Economics of the Environment, Natural Resources and Climate Change

Final reexam

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(3-hour closed book exam)

EXERCISE 1. Optimal natural resource extraction with pollution and endogenous exploration

Consider a model of the economy and the environment that uses the following notation:

$Y =$ production of final goods
$K =$ stock of man-made capital (physical and human)
$R =$ input of an exhaustible natural resource (raw material)
$S =$ reserve stock of the natural resource
$E =$ total emission of pollutant
$b =$ emission of pollutant per unit of raw material used in final goods production
$a =$ cost of extracting one unit of the natural resource (measured in units of the final good)
$Q =$ total exploration costs (measured in units of the final good)
$C =$ consumption of final goods
$D =$ discovery of new reserves of the natural resource
$I =$ investment in produced capital
$U =$ lifetime utility of the representative consumer
$u =$ flow of utility from consumption of final goods
$\rho =$ rate of time preference
$t =$ time (treated as a continuous variable)

The pollution from the use of raw materials generates disutility for consumers. The lifetime utility of the representative consumer at time zero is therefore given as

$$U_0 = \int_0^\infty \left[u\left(C_t\right) - E_t \right] e^{-\rho t} dt, \qquad u' > 0, \quad u'' < 0, \quad \rho > 0. \tag{1}$$

In the following, all variables except the constant parameters a, b and ρ will be understood to be functions of time, so for convenience we will generally skip the time subscripts.

The production of final goods is given by the production function

$$Y = F\left(K, R\right), \qquad F_K \equiv \frac{\partial F}{\partial K} > 0, \quad F_R \equiv \frac{\partial F}{\partial R} > 0. \tag{2}$$

Pollution is caused by the transformation of the raw material in the process of production. The use of one unit of raw material generates an emission of b units of the pollutant, so total emissions are

$$E = bR, \qquad b \text{ constant.} \tag{3}$$

In each period there is exploration for the discovery of new reserves of the natural resource. The total exploration cost (measured in units of the final good) of discovering the quantity D of new reserves is given by the following exploration cost function:

$$Q = Q\left(D, S\right), \qquad Q_D \equiv \frac{\partial Q}{\partial D} > 0, \quad Q_S \equiv \frac{\partial Q}{\partial S} < 0. \tag{4}$$

The functions F and Q are assumed to have properties which ensure that the first-order conditions for the optimization problems considered below indeed identify maxima.

The total cost of raw material production (measured in units of final goods) is aR, where the constant a is the cost of extracting one unit of the natural resource from the existing stock of reserves. Hence the economy's aggregate resource constraint is

$$Y = C + I + Q + aR, \qquad a \text{ constant}, \quad a > 0. \tag{5}$$

We will abstract from depreciation, so the net investment in man-made capital is equal to the gross investment I. Hence the change over time in the capital stock is

$$\dot{K} \equiv \frac{dK}{dt} = I. \tag{6}$$

The change over time in the natural resource stock is

$$\dot{S} \equiv \frac{dS}{dt} = D - R. \tag{7}$$

The initial stocks of man-made and natural capital (K_0 and S_0) are predetermined.

Our first task is to characterize the first-best optimal allocation of resources that would be chosen by a benevolent social planner who maximizes the utility function (1) subject to the constraints implied by eqs. (2) through (7), taking K_0 and S_0 as given.

Question 1.1: Give a brief motivation for the specification of the exploration cost function (4). (Hint: How do you motivate the assumptions on the signs of the partial derivatives?)

Answer to Question 1.1: It is natural to assume that it requires a greater exploration effort and hence a larger total exploration cost to discover a larger amount of reserves. This motivates the assumption that $Q_D > 0.$ The assumption that $Q_S < 0$ might be motivated as follows: A larger stock of reserves (S) indicates greater abundance of the natural resource. The greater abundance makes it easier and hence less costly to locate new reserves. However, this relationship between S and Q is not obvious, since current reserves depend in part on past exploration. As exploration continues over time, it may become more difficult to locate new reserves, implying that a higher S (resulting from past exploration) may be associated with higher current exploration cost. *(Note: It is not considered an error if the student does not discuss this ambiguity of the sign of Q_S).*

Question 1.2: Show that the current-value Hamiltonian corresponding to the social planner's problem may be written as

$$H = u(C) - bR + \mu\left[F(K,R) - C - Q(D,S) - aR\right] + \lambda(D - R), \tag{8}$$

where μ is the shadow value of K, and λ is the shadow value of S. What are the control variables and what are the state variables in the social planner's optimal control problem?

Answer to Question 1.2: The control variables are *C, R,* and *D*, and the state variables are *K* and *S*. The current-value Hamiltonian takes the general form

$$H = u(C) - E + \mu \dot{K} + \lambda \dot{S} \tag{i}$$

From (2), (4), (5) and (6) we have

$$\begin{aligned} \dot{K} &= I = Y - C - Q - aR \\ &= F(K,R) - C - Q(D,S) - aR \end{aligned} \tag{ii}$$

Inserting (ii), (3), and (7) into (i), we end up with (8).

Question 1.3: Derive the first-order conditions for the solution to the social planner's optimal control problem.

Answer to Question 1.3: From (8) we get the first-order conditions

$$\frac{\partial H}{\partial C} = 0 \quad \Rightarrow \quad u'(C) = \mu \tag{iii}$$

$$\frac{\partial H}{\partial R} = 0 \quad \Rightarrow \quad \mu(F_R - a) - b = \lambda \tag{iv}$$

$$\frac{\partial H}{\partial D} = 0 \quad \Rightarrow \quad \lambda = \mu Q_D \tag{v}$$

$$\dot{\mu} = \rho\mu - \frac{\partial H}{\partial K} \quad \Rightarrow \quad \dot{\mu} = \mu(\rho - F_K) \tag{vi}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial S} \quad \Rightarrow \quad \rho\lambda + \mu Q_S \tag{vii}$$

In addition, an optimal solution must satisfy the transversality conditions

$$\lim_{t\to\infty} e^{-\rho t}\mu_t K_t = 0, \qquad \lim_{t\to\infty} e^{-\rho t}\lambda_t S_t = 0. \tag{viii}$$

(Note: Since the transversality conditions in (viii) will not be used later, it is not considered an error if they are left out of the answer to this question).

Question 1.4: Show that the first-order conditions for the solution to the social planner's optimal control problem imply that

$$F_R = a + Q_D + MEC, \qquad MEC \equiv \frac{b}{\mu}. \tag{9}$$

Give an economic interpretation of eq. (9) and explain the economic intuition behind it.

Answer to Question 1.4: Inserting (v) in (iv), we get

$$\mu\left(F_R - a\right) - b = \mu Q_D \tag{ix}$$

Dividing by μ on both sides of (ix) and rearranging, we obtain (9), using the definition of *MEC* stated in that equation. The variable *MEC* is the marginal external cost of raw materials use measured in units of the consumption good *C*, since it follows from (1) and (3) that the pollution caused by the use of an additional unit of raw material reduces consumer welfare by *b* units of utility which is equivalent to the welfare cost of a b/μ unit drop in consumption, since we know from (iii) that the marginal utility of consumption is μ. The left-hand side of (9) is the marginal social benefit of raw materials use, given by the rise in final output F_R generated by the use of an extra unit of raw material in production. The right-hand side of (9) is the marginal social cost of raw material use, consisting of the marginal extraction cost *a* plus the cost Q_D of replacing the used-up raw material by new discoveries plus the marginal external cost *MEC* of raw material use. Thus eq. (9) says that the marginal social benefit from using raw materials should equal the marginal social cost.

Question 1.5: Show that the first-order conditions for the solution to the social planner's problem also imply the following condition for an optimal exploitation of the natural resource, where *MEC* is defined in (9):

$$\dot{F}_R - Q_S + \rho MEC = \left(F_R - a\right) F_K. \tag{10}$$

Give an economic interpretation of eq. (10) and explain the economic intuition behind it.

Answer to Question 1.5: Differentiating both sides of (iv) with respect to time and remembering that *a* and *b* are constants, we get

$$\dot{\mu}\left(F_R - a\right) + \mu \dot{F}_R = \dot{\lambda} \tag{x}$$

Substitution of (vi) and (vii) into (x) yields

$$\mu\left(\rho - F_K\right)\left(F_R - a\right) + \mu \dot{F}_R = \rho\lambda + \mu Q_S \tag{xi}$$

Next we use (iv) to eliminate λ from (xi) and divide through by μ to get

$$\mu(\rho - F_K)(F_R - a) + \mu \dot{F}_R = \rho\left[\mu(F_R - a) - b\right] + \mu Q_S \Leftrightarrow$$

$$\rho(F_R - a) - F_K(F_R - a) + \dot{F}_R = \rho(F_R - a) - \rho\frac{b}{\mu} + Q_S \Leftrightarrow$$

$$\dot{F}_R + \rho MEC - Q_S = F_K(F_R - a)$$

where we have used the definition $MEC \equiv b/\mu.$ We see that the bottom line above is identical to (10). The left-hand side of (10) is the gain "tomorrow" from leaving one extra unit of the raw material in the underground "today". This gain consists of the rise $\dot{F}_R$ in the marginal productivity of the raw material between today and tomorrow plus the gain ρMEC from the postponement of the environmental cost of extracting an extra unit of raw material plus the fall $-Q_S$ in tomorrow's discovery costs as tomorrow's reserve stock becomes larger when extraction is postponed. Note that the consumer's gain ρMEC from the postponement of pollution increases with the rate of time preference ρ, reflecting that the consumer's valuation of current welfare relative to future welfare is higher the greater the value of ρ. The right-hand side of (10) is the gain "tomorrow" from extracting an extra unit of raw material today and investing the resource rent $F_R - a$ in man-made capital yielding a marginal return F_K equal to capital's marginal product. When (10) is satisfied, the intertemporal allocation of raw materials extraction is therefore socially optimal, since society can gain nothing from postponing or accelerating the extraction of an extra unit of the natural resource. Dividing through by the resource rent $F_R - a$, we can also write (10) as

$$\frac{\dot{F}_R + \rho MEC - Q_S}{F_R - a} = F_K \qquad \text{(xii)}$$

The left-hand side of (xii) is the marginal rate of return on investment in natural capital (where the investment takes the form of leaving an extra unit of the resource in the ground today), and the right-hand side is the marginal rate of return on investment in man-made capital. In optimum the two rates of return must be identical for society to have an optimal portfolio of natural and man-made capital. This is a Hotelling Rule for socially optimal natural resource management, modified to account for pollution and for endogenous exploration costs (*end of answer to Question 1.5*).

We will now consider the resource allocation that will materialize in a market economy with private property and perfect competition in all markets. We start by focusing on the production of raw materials, and we assume that the reserves of natural resources are owned by private mining firms which extract materials from the existing reserve stock and engage in exploration for new reserves. The market value of the representative mining firm at time zero is denoted by V_0^R, and the firm's payout of net dividends in the future period t is denoted by D_t^R. The market value of the firm is the present value of the future net dividends paid out to its owners, that is

$$V_0^R = \int_0^\infty D_t^R e^{-\int_0^t r_s ds} dt, \tag{11}$$

where r is the real market interest rate which may vary over time. The market price of the raw material (measured relative to the price of final goods) is p, and the mining firm must pay an extraction tax τ for each unit of raw material extracted. Both p and τ may also vary over time. The net dividend paid out by the mining firm in period *t* is therefore given by

$$D_t^R = \left(p_t - a - \tau_t\right) R_t - Q\left(D_t, S_t\right). \tag{12}$$

The mining firm chooses its rate of raw material extraction R and its rate of discovery D so as to maximize its market value (11) subject to (12), taking the price p and the tax rate τ as given, and accounting for the stock-flow relationship (7) between its current rates of extraction and discovery and its remaining reserve stock of the resource.

Question 1.6: Set up the current-value Hamiltonian for the mining firm's optimal control problem and derive the first-order conditions for the solution to its problem (you may denote the shadow price of the reserve stock by λ^R, and for convenience you may skip the time subscripts).

Answer to Question 1.6: The current-value Hamiltonian H^R for the mining company's problem has the form

$$H^R = D^R + \lambda^R \dot{S} \tag{xiii}$$

Inserting (7) and (12) in (xiii), we get

$$H^R = \left(p - a - \tau\right) R - Q\left(D, S\right) + \lambda^R \left(D - R\right) \tag{xiv}$$

Question 1.7: Show that the mining firm's first-order conditions imply that

$$\dot{p} - \dot{\tau} - Q_S = r\left(p - a - \tau\right). \tag{13}$$

Give an economic interpretation of eq. (13) and explain the economic intuition behind it.

Answer to Question 1.7: The mining firm's control variables are *R* and *D*, while its state variable is *S*. From (xiv) we therefore obtain the first-order conditions

$$\frac{\partial H^R}{\partial R} = 0 \quad \Rightarrow \quad p - a - \tau = \lambda^R \tag{xv}$$

$$\frac{\partial H^R}{\partial D} = 0 \quad \Rightarrow \quad \lambda^R = Q_D \tag{xvi}$$

$$\dot{\lambda}^R = r\lambda^R - \frac{\partial H^R}{\partial S} \qquad \Rightarrow \qquad \dot{\lambda}^R = r\lambda^R + Q_S \tag{xvii}$$

plus the following transversality condition:

$$\lim_{t\to\infty} e^{-\int_0^t r_s ds} \lambda_t^R S_t = 0 \tag{xviii}$$

(*Note: It is no error if the transversality condition (xviii) is omitted).* Differentiating (xv) with respect to time and inserting (xvii) into the resulting expression, we find

$$\dot{p} - \dot{\tau} = \dot{\lambda}^R \quad \Rightarrow$$

$$\dot{p} - \dot{\tau} = r\lambda^R + Q_S \tag{xix}$$

By substituting (xv) into (xix) and rearranging, we obtain eq. (13) which is a variant of the Hotelling Rule for privately optimal resource extraction. The left-hand side of (13) is the mining company's net gain "tomorrow" from leaving an extra unit of the raw material in the ground today, This net gain consists of the tax-adjusted increase in the resource rent between today and tomorrow ($\dot{p} - \dot{\tau}$) plus the fall $-Q_S$ in tomorrow's discovery costs as tomorrow's reserve stock becomes larger when extraction is postponed. The right-hand side of (13) is the extra dividend that can be paid out to the mining company's owners tomorrow if an extra unit is extracted and sold today, thereby generating an additional tax-adjusted resource rent $p - a - \tau$ which can be invested in the capital market at the interest rate *r*. The Hotelling Rule (13) therefore says that a value-maximizing mining firm will be indifferent between extracting an extra unit today or postponing extraction until tomorrow (*end of answer to Question 1.7*).

Consider next the representative firm producing the final good Y which we use as our numeraire good, thus setting its price equal to 1. The firm's market value at time zero (V_0^Y) is the present value of its future net dividends,

$$V_0^Y = \int_0^\infty D_t^Y e^{-\int_0^t r_s ds} dt, \tag{14}$$

where D_t^Y is the net dividend paid out in the future period t. The output of the final goods firm is given by the production function (2), so the net dividend paid out by the firm is

$$D_t^Y = F(K_t, R_t) - p_t R_t - I_t. \tag{15}$$

The final goods firm chooses R and I with the purpose of maximizing its market value (14) subject to (15), accounting for the stock-flow relationship (6) between its investment and the change in its capital stock.

Question 1.8: Set up the current-value Hamiltonian for the optimal control problem of the final goods firm (you may denote the shadow price of its capital stock by μ^Y, and for convenience you may skip the time subscripts). Show that the first-order conditions for the solution to the problem of the final goods firm imply that

$$F_R = p, \tag{16}$$

$$F_K = r. \tag{17}$$

Give an economic interpretation of these results.

Answer to Question 1.8: The current-value Hamiltonian H^Y for the final goods firm takes the general form

$$H^Y = D^Y + \mu^Y \dot{K} \tag{xx}$$

Inserting (6) and (15) in (xx), we get

$$H^Y = F\left(K, R\right) - pR - I + \mu^Y I \tag{xxi}$$

The control variables of the final goods firm are R and I, and its state variable is K. Hence we obtain the following first-order conditions from (xxi):

$$\frac{\partial H^Y}{\partial R} = 0 \quad \Rightarrow \quad F_R = p \tag{xxii}$$

$$\frac{\partial H^Y}{\partial I} = 0 \quad \Rightarrow \quad \mu^Y = 1 \tag{xxiii}$$

$$\dot{\mu}^Y = r\mu^Y - \frac{\partial H^Y}{\partial K} \quad \Rightarrow \quad \dot{\mu}^Y = r\mu^Y - F_K \tag{xxiv}$$

In addition, an optimal solution to the problem of the final goods firm must satisfy the following transversality condition (*Note: It is no error if this transversality condition is omitted)*:

$$\lim_{t\to\infty} e^{-\int_0^t r_s ds} \mu_t^Y K_t = 0 \tag{xxv}$$

The result (16) is simply the first-order condition for optimal raw material use stated in (xxii). From (xxiii) it follows that $\dot{\mu}^Y = 0$. Inserting this along with (xxiii) into (xxiv), we obtain the result in (17). Eqs. (16) and (17) are standard conditions for profix maximization stating that the marginal products of the two production factors R and K must be equal to their respective factor prices.

Question 1.9: Use your findings in Question 1.6, 1.7 and 1.8 to derive an expression for the optimal environmental tax rate τ which will ensure that the use of natural resources in the market economy will satisfy the social optimality conditions (9) and (10). Explain the economic intuition for your result.

Answer to Question 1.9: From the mining firm's optimum conditions (xv) and (xvi) it follows that

$$p = a + Q_D + \tau \tag{xxvi}$$

Substituting the optimum condition (xxii) for the final goods firm into (xxvi), we get

$$F_R = a + Q_D + \tau \tag{xxvii}$$

If the tax on extraction of raw material is set at the rate,

$$\tau = \frac{b}{\mu} \equiv MEC, \tag{xxviii}$$

we see that the private optimum condition (xxvii) determining the use of raw materials in the market economy will coincide with the social optimum condition (9).

We now need to show that if the extraction tax is set at the rate (xxviii), the Hotelling Rule (10) for a social optimum will also be obeyed. To demonstrate this, we note from (xv) and (xix) that

$$\dot{p} - \dot{\tau} = r\left(p - a - \tau\right) + Q_S \tag{xxix}$$

We also see from (xxii) that

$$p = F_R \quad \Rightarrow \quad \dot{p} = \dot{F}_R \tag{xxx}$$

Inserting the results in (xxx) into (xxix) along with our earlier result $r = F_K$, we get

$$\dot{F}_R - \dot{\tau} - Q_S = F_K\left(F_R - a - \tau\right) \tag{xxxi}$$

Differentiating the expression for the tax rate stated in (xxviii) with respect to time, remembering that b is a constant, and using the first-order condition (vi), we find

$$\dot{\tau} = \frac{-b\,\dot{\mu}}{\mu^2} = \frac{-b\;\mu\left(\rho - F_K\right)}{\mu^2} = \frac{b}{\mu}\left(F_K - \rho\right) \quad \Rightarrow$$

$$\dot{\tau} = \left(F_K - \rho\right)MEC \tag{xxxii}$$

We can now substitute (xxviii) and (xxxii) into (xxxi) to get

$$\dot{F}_R - (F_K - \rho) MEC - Q_S = F_K (F_R - a - MEC) \quad \Leftrightarrow$$

$$\dot{F}_R + \rho MEC - Q_S = F_K (F_R - a) \qquad \text{(xxxiii)}$$

Equation (xxxiii) is seen to be identical to the social optimum condition (10).

We have now proved that an extraction tax rate set at the level specified in (xxviii) will satisfy the conditions (9) and (10) for a socially optimal use of natural resources. The tax rate (xxviii) is the Pigouvian tax which fully internalizes the marginal environmental cost of the pollution generated by the extraction of raw materials. When this tax rate is added to the market price of raw materials, mining firms are confronted with the full marginal social cost of extracting an extra unit of the natural resource.

EXERCISE 2. The Hartwick Rule and sustainable development

(Note: The questions in this exercise may be answered without any use of math. However, you are welcome to use math to the extent that you find it convenient).

Question 2.1: Explain the Hartwick Rule for natural resource management.

Answer to Question 2.1: The Hartwick Rule states that all of the resource rents from extraction of exhaustible natural resources should be invested in man-made capital. This means that none of the gross rent from extraction (the rent before deduction for extraction costs) may be used for consumption. Only the return to the man-made capital generated through the investment of natural resource rents may be consumed.

If the total output of final goods is given by a production function like (2) with constant returns to scale and the total cost of natural resource extraction is aR, where a is the constant marginal extraction cost, the Hartwick Rule requires that

$$\dot{K} = (F_R - a) R \qquad \text{(xxxiv)}$$

The term $F_R - a$ on the right-hand side of (xxxiv) is the marginal resource rent which is equal to the average resource rent under constant returns to scale, so the term $(F_R - a)R$ is the economy's aggregate resource rent. (*Note: It is not considered an error if the student's answer to the question does not include an equation like (xxxiv)*).

Question 2.2: Explain how the Hartwick Rule relates to the concept of sustainable development.

In Environmental Economics sustainable development is often defined as a development path where the utility of the representative individual is non-declining over time. If utility per period (u)

depends only on consumption, i.e., $u = u(C)$, sustainable development thus requires that consumption be non-declining. It can be shown that, in an economy without technical progress and a constant-returns production function of the form (2), the Hartwick Rule combined with the Hotelling Rule for optimal natural resource extraction will ensure sustainable development in the sense that consumption (and hence utility) is constant over time.

The formal proof of this proposition goes as follows: The production function (2) implies that

$$\dot{Y} = F_K \dot{K} + F_R \dot{R} \qquad \text{(xxxv)}$$

and the Hotelling Rule requires that

$$\dot{F}_R = F_K (F_R - a) \qquad \text{(xxxvi)}$$

Inserting the Hartwick Rule (xxxiv) and the Hotelling Rule (xxxvi) in (xxxv), we get

$$\begin{aligned} \dot{Y} &= F_K \dot{K} + F_R \dot{R} \\ &= F_K (F_R - a) R + F_R \dot{R} \\ &= \dot{F}_R R + F_R \dot{R} \quad \Rightarrow \end{aligned}$$

$$\dot{Y} = \frac{d(F_R R)}{dt} \qquad \text{(xxxvii)}$$

Since the Hartwick Rule (xxxiv) can be rearranged to give $F_R R = \dot{K} + aR$, we can rewrite (xxxvii) as

$$\dot{Y} = \frac{d\left(\dot{K} + aR\right)}{dt} \qquad \text{(xxxviii)}$$

But the economy's aggregate resource constraint also implies that

$$Y = C + \dot{K} + aR \quad \Rightarrow$$

$$\dot{Y} = \dot{C} + \frac{d\left(\dot{K} + aR\right)}{dt} \qquad \text{(xxxix)}$$

Obvisouly (xxxviii) and (xxxix) cannot be satisfied at the same time unless

$$\dot{C} = 0 \qquad \text{(xxxx)}$$

Equation (40) shows that adherence to the Hartwick Rule will ensure sustainable development in the sense that consumption (and hence, by assumption, utility) will be non-declining over time. *(Note: Only an outstanding student will be able to reproduce the formal mathematical proof given above, so it is not considered an error if this part of the answer to Question 2.2 is omitted).*